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An Inverse Optimal Stopping Problem for Diffusion Processes

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Abstract. Let X be a one-dimensional diffusion and let g be a real-valued function depending on time and the value of X. This article analyzes the inverse optimal stopping problem of finding a time-dependent real-valued function π depending only on time such that a given stopping time $\tau^\star$ is a solution of the stopping problem $\sup_\tau \mathbb{E}[g(\tau, X_\tau) + \pi(\tau)]$.

Under regularity and monotonicity conditions, there exists such a transfer π if and only if $\tau^\star$ is the first time when X exceeds a time-dependent barrier b. We prove uniqueness of the solution π and derive a closed form representation. The representation is based on an auxiliary process that is a version of the original diffusion X reflected at b towards the continuation region. The results lead to a new integral equation characterizing the stopping boundary b of the stopping problem $\sup_\tau \mathbb{E}[g(\tau, X_\tau)]$.

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1. Introduction

Optimal stopping is omnipresent in applications of dynamic optimization in economics, statistics, and finance. Examples are the optimal exercise timing of options, when to stop searching, and the quickest detection problem. One method to solve optimal stopping problems in a Markovian framework is to identify the stopping region. Optimal stopping times are then given as first hitting times of the stopping region.

Many applications of optimal stopping naturally lead to the question of how to change a payoff such that a given stopping rule becomes optimal. Mathematically, this *inverse* optimal stopping problem consists of modifying the payoff of a stopping problem in such a way that it is optimal to stop at the first time when a *given* set is hit. In many economic applications informational constraints furthermore restrict the set of admissible modifications to the addition of a time-dependent function to the original payoff, that is, transfers that are independent of the realization of the process.

To fix ideas, consider the continuous-time, finite horizon optimal stopping problem

$$\sup_{\tau \in \mathcal{T}} \mathbb{E}[g(\tau, X_\tau)],$$

where $\mathcal{T}$ is the set of stopping times with values in $[0, T]$, X is a one-dimensional diffusion, and g is a smooth payoff function. Throughout the paper, we suppose that a so-called single crossing condition is satisfied. It requires that the expected gain of waiting an infinitesimal amount of time is nonincreasing in the value of the process X. Formally, we suppose that the function

$$x \mapsto \lim_{h \searrow 0} \frac{1}{h} \mathbb{E}[g(t+h, X^{t,x}_{t+h}) - g(t,x)] = (\partial_t + \mathcal{L}) g(t,x) \tag{1}$$

is nonincreasing, where $\mathcal{L}$ denotes the generator of the diffusion X. A deterministic function π: $[0,T] \to \mathbb{R}$ is called a transfer. We say that a set $A \subset [0,T] \times \mathbb{R}$ is implemented by a transfer π if the first time τ_A when X hits A is optimal in the stopping problem with payoff $g + \pi$, that is, if

$$\tau_A \in \arg\sup_{\tau \in \mathcal{T}} \mathbb{E}[g(\tau, X_\tau) + \pi(\tau)]. \tag{2}$$

1.1. Applications

Inverse optimal stopping problems play an important role in different economic situations. For example, they can be used to formulate dynamic principal-agent problems: There is an agent who *privately* observes the stochastic process X and aims at maximizing her expected payoff $\sup_{\tau\in\mathcal{T}}\mathbb{E}[g(\tau,X_\tau)]$ from stopping the process. The principal observes the stopping decision of the agent, but not the realization of the process. She aims at inducing the agent to take a particular stopping decision given by the hitting time τ_A. To influence the agent's stopping decision, the principal commits to a transfer π, which specifies a payment that is due at the moment when the agent stops. The principal needs to construct the transfer π in such a way that τ_A becomes optimal in the modified stopping problem $\sup_{\tau\in\mathcal{T}}\mathbb{E}[g(\tau,X_\tau)+\pi(\tau)]$. For example, the agent could be a firm that has developed a new technology and now has to decide when to introduce it to the market place. Over time the firm observes private signals regarding the demand. The principal is a social planner who also takes the consumer surplus of the new technology into account and hence prefers a different stopping decision than the firm. The inverse optimal stopping problem analyzes the question how the planner can align the preferences of the firm by subsidizing the market entry through a transfer (see Section 2.1 for a specific example).

Other economic examples of inverse optimal stopping problems are the design of unemployment benefits (McCall [27], Hopenhayn and Nicolini [18]), the structuring of management compensation, the sale of irreversible investment options (Board [3]), as well as the inference of deliberation costs in search theory (Drugowitsch et al. [9], Fudenberg et al. [12]). Section 2 presents two more specific examples. For further economic examples and applications to revenue management we refer to Kruse and Strack [24], where inverse optimal stopping problems have been introduced in a discrete-time framework.

1.2. Results

The main result (Theorem 11) states that all cutoff regions $A=\{(t,x)\mid x\ge b(t)\}$ are implementable provided that the boundary b is càdlàg and has summable downwards jumps. Furthermore, we show that the solution π implementing the cutoff region $A=\{(t,x)\mid x\ge b(t)\}$ admits the following closed form representation

$$\pi(t)=\mathbb{E}\left[\int_t^T(\partial_t+\mathcal{L})g(s,\tilde{X}_s^{t,b(t)})\,\mathrm{d}s\right]. \tag{3}$$

Here $(\tilde{X}_s^{t,b(t)})_{s\ge t}$ denotes the unique process starting on the barrier $b(t)$ at time t, which results from reflecting the original process X at the barrier $(b(s))_{s\in[t,T]}$ away from A.

As shown in Kotlow [23], Jacka and Lynn [20] and Villeneuve [44] the single crossing condition (1) (or a weaker version of it) ensures that the stopping region in stopping problems of the form $v(t,x)=\sup_{\tau\in\mathcal{T}_{t,T}}\mathbb{E}[g(\tau,X_\tau^{t,x})]$ is of cutoff type, that is, there exists a barrier $b\colon[0,T]\to\mathbb{R}$ such that $x\ge b(t)$ if and only if $v(t,x)=g(t,x)$. In Proposition 7 we show that this result translates to implementable regions. We introduce the notion of *strict* implementability for sets $A\subset[0,T]\times\mathbb{R}$, where we additionally demand that A coincides with the stopping region of the problem (2). Proposition 7 states that under the single crossing condition, only cutoff regions are strictly implementable. Furthermore, we show that if the monotonicity in Equation (1) is strict, then cutoff regions with a càdlàg barrier with summable downward jumps are strictly implementable (Corollary 12). In this way the following characterization of strictly implementable regions holds up the assumption of right continuity and summable downward jumps: A region is strictly implementable if and only if it is of cutoff type.

Furthermore, the transfer implementing a cutoff region is unique up to an additive constant (Theorem 15). This result leads to a new characterization of optimal stopping boundaries (Corollary 16). If the first hitting time τ_A of a set A is optimal in the stopping problem $\sup_{\tau\in\mathcal{T}}\mathbb{E}[g(\tau,X_\tau)]$ then A is implemented by the zero transfer. Uniqueness of the transfer implies that

$$\mathbb{E}\left[\int_t^T(\partial_t+\mathcal{L})g(s,\tilde{X}_s^{t,b(t)})\,\mathrm{d}s\right]=0 \tag{4}$$

for all $t\in[0,T]$. Remarkably, the nonlinear integral Equation (4) is not only necessary but also sufficient for optimality. In Section 6 we discuss the relation to the integral equation derived in Kim [22], Jacka [19], and Carr et al. [5] (see also Peskir and Shiryaev [38]).

The paper is organized as follows. Section 2 presents specific examples of inverse optimal stopping problems. In Section 3 we set up the model and introduce the notion of implementability. In Section 4 we show that only cutoff regions are strictly implementable. Section 5 is devoted to the converse implication. First we introduce reflected processes and formally derive the representation (3) of the transfer (Section 5.1). Section 5.2 contains the main results about implementability of cutoff regions. In Section 5.3 we present the main properties of the transfer and in Section 5.4 we provide the uniqueness result. In Section 6 we derive and discuss the integral equation (4).

2. Motivating Examples

2.1. Providing Incentives for Investment in a Project of Unknown Profitability

A single agent (or firm) can invest into a project of unknown value $\theta \in \mathbb{R}$. The value (or discounted expected future return) of the project $\theta \in \mathbb{R}$ is normally distributed with mean X_0 and variance σ_0^2. The agent learns about the project's value over time by observing a signal (or payoff) (Z_t), which is a Brownian motion (W_t) (independent of θ) plus drift equal to the true return of the project

$$dZ_t = \theta\, dt + dW_t.$$

When the agent invests into the project at time τ he receives its value discounted by the time at which he invested $e^{-r\tau}\theta$. The agent's problem is to find a stopping time adapted to $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ (the natural filtration of Z), which solves $\sup_\tau \mathbb{E}[e^{-r\tau}\theta]$. If we denote by $X_t = \mathbb{E}[\theta \mid \mathcal{F}_t]$ the posterior expected value that the agent assigns to the project, the law of iterated expectations implies that this problem is equivalent to

$$\sup_\tau \mathbb{E}[e^{-r\tau} X_\tau].$$

It is well known (cf. Theorem 10.1 in Liptser and Shiryaev [26]) that after seeing the signal $(Z_s)_{s\leq t}$ the agents posterior belief about the value of the project is normally distributed with variance $\sigma_t^2 = 1/(\sigma_0^{-2} + t)$ and mean $X_t = \sigma_t^2(X_0\sigma_0^{-2} + Z_t)$, and furthermore that there exists a Brownian motion B_t (in the filtration $\mathbb{F}$) such that

$$dX_t = \sigma_t^2\, dB_t.$$

Hence, the agent's learning and investment problem is equivalent to the problem of stopping the diffusion X. As the problem is Markovian in (t, X) and the returns from waiting to invest are decreasing in X_t it follows that the optimal solution is to invest once the expected value of the project X_t exceeds a time-dependent threshold $b^0(\cdot)$[1]

$$\tau = \inf\{t: X_t \geq b^0(t)\}.$$

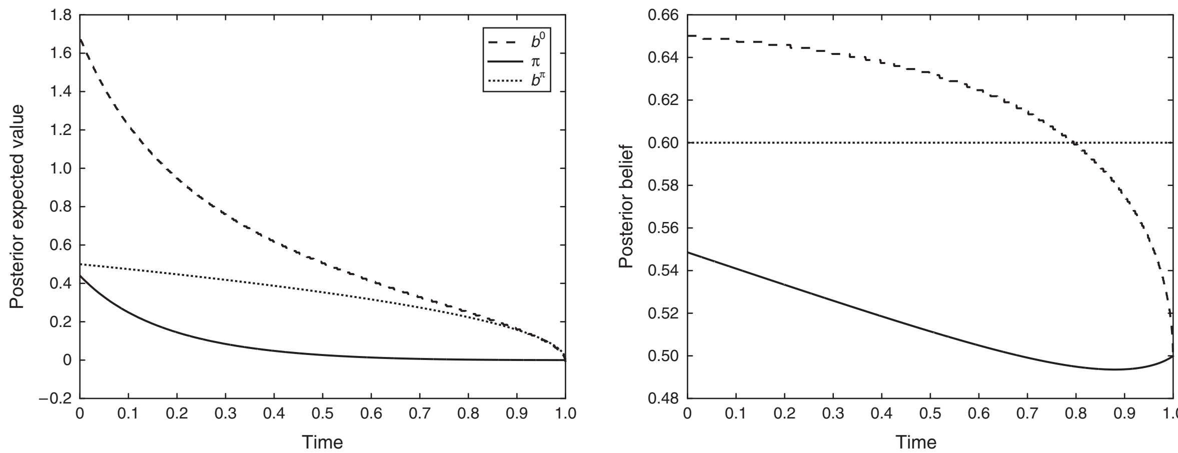
The decision to invest into a project might, for example, correspond to bringing a new product to the market. In many such investment situations the incentive of the firm to invest is not aligned with the incentives of society. For example, a pharmaceutical firm that takes an investment decision based upon the profitability of a treatment, ignoring consumer surplus generated from the availability of medicine, will invest too late and in too few treatments. To mitigate this inefficiency the government could use subsidies (and taxes) on new projects that depend on the time the firm invests and brings the project to the market. For example, in Figure 1 the dashed line shows the investment threshold $b^0: [0,T] \to \mathbb{R}$ at which the firm invests without a transfer. Suppose that the government wants the firm to invest earlier, for example, at the first time when the firm's belief about the investment value X exceeds the barrier $b^\pi: [0,T] \to \mathbb{R}, b^\pi(t) = 0.5\sqrt{T-t}$. In Section 5 we show that this is possible for the government by using a transfer $\pi: [0,T] \to \mathbb{R}$. The solid line in the left-hand side of Figure 1 depicts such a transfer.

2.2. Quickest Change Point Detection in a Principal-Agent Framework

Quickest detection problems play a prominent role in mathematical statistics and are a key ingredient in a number of models in the applied sciences such as quality control, epidemiology, and geology (see, e.g., Shiryaev [42, Chapter IV], Peskir and Shiryaev [38, Chapter IV, Section 22], and Müller and Siegmund [29] for historical accounts on the problem formulations and specific applications). As an economic motivation, consider a venture capitalist and an entrepreneur. The entrepreneur observes an informative signal about whether it is still profitable to run the firm or not. This information is often unobservable to the venture capitalist as he possesses no knowledge of the specific market. The venture capitalist who finances the firm wants to stop operations once he is 60% sure that the firm became unprofitable. The entrepreneur might prefer running the firm much longer as doing so yields private benefits to him. An important question in the venture capital industry is how to design compensation schemes that align the interests of the entrepreneur with those of the venture capitalist.

We consider here a variant of the quickest detection problem of a Wiener process from a principal-agent perspective. For the formulation of the single-agent quickest detection problem we follow closely (Gapeev and Peskir [14]). There is an agent observing on the finite time interval $[0,T]$ the path of a one-dimensional Brownian motion X, which changes its drift from 0 to $\mu \neq 0$ at some random time θ. The random time θ is independent of X and is exponentially distributed with parameter $\lambda \in (0,\infty)$. The agent does not observe θ, but has to infer information about θ from the continuous observation of X. The agent's goal is to find a stopping time of X

Figure 1. Left: The dashed line depicts the optimal investment threshold (stopping barrier) b^0 in the setting of Example 2.1 when there is no interference by the government. To incentivize the firm to invest at the first time when the firm's belief about the investment value X exceeds the barrier b^π: $[0,T]\to\mathbb{R}$, $b^\pi(t)=0.5\sqrt{T-t}$, the government can use the transfer π: $[0,T]\to\mathbb{R}$ given by the solid line. The figure is generated using the parameters $r=1$, $\sigma_0^2=4$ and $T=1$. Right: the dashed line shows the optimal stopping barrier b^0 of the stopping problem (5) of Example 2.2. The solid line depicts a transfer π: $[0,T]\to\mathbb{R}$ that induces the agent to stop at the first time when the posterior belief p exceeds the level 60%. For the figure, the parameters $c=\mu=1$ and $T=0.5$ are used.



that is as close to θ as possible. More formally, for $c\in(0,\infty)$ the agent aims at finding a $[0,T]$-valued stopping time τ with respect to the filtration $\mathcal{F}^X$ generated by X that attains the minimum in

$$\inf_{0\le\tau\le T}(\mathbb{P}[\tau\le\theta]+c\mathbb{E}[(\tau-\theta)^+]).$$

As shown in Gapeev and Peskir [14], this is equivalent to solving the stopping problem

$$\inf_{0\le\tau\le T}\mathbb{E}\left[1-p_\tau+c\int_0^\tau p_t\,\mathrm{d}t\right]=1-\sup_{0\le\tau\le T}\mathbb{E}\left[\int_0^\tau \lambda-(c+\lambda)p_t\,\mathrm{d}t\right], \tag{5}$$

where the process p satisfies for all $t\in[0,T]$ that

$$dp_t=\lambda(1-p_t)\,\mathrm{d}t+\mu p_t(1-p_t)\,\mathrm{d}W_t,\quad p_0=0$$

and where W is a standard Brownian motion. The process p satisfies for all $t\in[0,T]$ that $p_t=\mathbb{P}[\theta\le t\mid\mathcal{F}_t^X]$ and thus describes at each time $t\in[0,T]$ the posterior belief whether θ already occurred. Now suppose that there is a principal who does neither observe X nor θ, but is notified at the moment when the agent stops. The principal's goal is to construct a transfer π: $[0,T]\to\mathbb{R}$ to the agent such that the principal is notified at the first time before T when the posterior belief p exceeds a threshold level of, say, 60%. It follows from the results in Section 5 that this is possible (note in particular that the flow payoff $p\mapsto\lambda-(c+\lambda)p$ in (5) is a decreasing function, cf. Condition 4). The right-hand side of Figure 1 shows such a transfer π.

3. Problem Formulation

3.1. Dynamics

We consider optimal stopping problems with finite time horizon $T<\infty$. The underlying probability space $(\Omega,\mathcal{F},\mathbb{P})$ supports a one-dimensional Brownian motion W. Let $\mathbb{F}=(\mathcal{F}_t)_{t\in[0,T]}$ be the filtration generated by W satisfying the usual assumptions. We denote the set of $\mathbb{F}$-stopping times with values in $[0,T]$ by $\mathcal{T}$. For $t<T$ we refer to $\mathcal{T}_{t,T}$ as the subset of stopping times that take values in $[t,T]$. The process X follows the time-inhomogeneous diffusion dynamics

$$\mathrm{d}X_t=\mu(t,X_t)\,\mathrm{d}t+\sigma(t,X_t)\,\mathrm{d}W_t. \tag{6}$$

We denote by $\mathcal{L}=\mu\partial_x+\frac{1}{2}\sigma^2\partial_{xx}$ the infinitesimal generator of X. The coefficients μ,σ: $[0,T]\times\mathbb{R}\to\mathbb{R}$ are Borel measurable functions satisfying the following global Lipschitz and linear growth assumptions: There exists a positive constant L such that

$$|\mu(t,x)-\mu(t,y)|+|\sigma(t,x)-\sigma(t,y)|\le L|x-y|$$
$$\mu(t,x)^2+\sigma(t,x)^2\le L^2(1+x^2)$$

for all $t \in [0,T]$ and $x, y \in \mathbb{R}$. Under this assumption there exists a unique strong solution $(X_s^{t,x})_{s\geq t}$ to (6) for every initial condition $X_t^{t,x} = x$ (see, e.g., Karatzas and Shreve [21, Theorems 2.5 and 2.9]). Moreover, it follows that the comparison principle holds true (see, e.g., Karatzas and Shreve [21, Proposition 2.18]): the path of the process starting at a lower level $x \leq x'$ at time t is smaller than the path of the process starting in x' at all later times $s > t$

$$X_s^{t,x} \leq X_s^{t,x'} \quad \mathbb{P}\text{-}a.s. \tag{7}$$

3.2. Payoffs and Transfers

As long as the process X is not stopped there is a flow payoff f and at the time of stopping there is a terminal payoff g. The payoffs $f, g\colon [0,T]\times\mathbb{R} \to \mathbb{R}$ depend on time and the value of the process X. Formally, the expected payoff for using a stopping time $\tau \in \mathcal{T}_{t,T}$ equals

$$W(t,x,\tau) = \mathbb{E}\left[\int_t^\tau f(s, X_s^{t,x})\,\mathrm{d}s + g(\tau, X_\tau^{t,x})\right],$$

given that X starts in $x \in \mathbb{R}$ at time $t \in [0,T]$. We assume that the payoff function f is continuous and Lipschitz continuous in the x variable uniformly in t. Moreover, we suppose that $g \in C^{1,2}([0,T]\times\mathbb{R})$ and that the functions g and $(\partial_t + \mathcal{L})g$ are Lipschitz continuous in the x variable uniformly in t.

We will analyze how preferences over stopping times change if there is an additional payoff that only depends on time.

Definition 1. A measurable, bounded function $\pi\colon [0,T] \to \mathbb{R}$ is called a transfer.

We define the value function $v^\pi\colon [0,T]\times\mathbb{R} \to \mathbb{R}$ of the stopping problem with payoffs f and g and an additional transfer π by

$$v^\pi(t,x) = \sup_{\tau\in\mathcal{T}_{t,T}} (W(t,x,\tau) + \mathbb{E}[\pi(\tau)]). \tag{8}$$

Moreover, we introduce for every $t \in [0,T]$ the stopping region

$$D_t^\pi = \{x \in \mathbb{R} \mid v^\pi(t,x) = g(t,x) + \pi(t)\}.$$

3.3. Implementability

A measurable set $A \subset [0,T]\times\mathbb{R}$ is called *time-closed* if for each time $t \in [0,T]$ the slice $A_t = \{x \in \mathbb{R} \mid (t,x) \in A\}$ is a closed subset of $\mathbb{R}$. Let X start in $x \in \mathbb{R}$ at time $t \in [0,T]$. For a time-closed set A we introduce the first time when X hits A by

$$\tau_A^{t,x} = \inf\{s \geq t \mid X_s^{t,x} \in A_s\} \wedge T.$$

We now come to the definition of implementability.

Definition 2 (Implementability)**.** A time-closed set A is implemented by a transfer π if the stopping time $\tau_A^{t,x}$ is optimal in (8), that is, for every $t \in [0,T]$ and $x \in \mathbb{R}$

$$v^\pi(t,x) = W(t,x,\tau_A^{t,x}) + \mathbb{E}[\pi(\tau_A^{t,x})].$$

For a time-closed set A a necessary condition for implementability is that each slice A_t is included in the stopping region D_t^π. Indeed, let A be implemented by π and let $t \in [0,T]$ and $x \in A_t$. Then we have $\tau_A^{t,x} = t$. Since $\tau_A^{t,x}$ is optimal, this implies $v^\pi(t,x) = g(t,x) + \pi(t)$ and hence $x \in D_t^\pi$. Consequently, we have $A_t \subseteq D_t^\pi$.

Observe that the converse inclusion $D_t^\pi \subseteq A_t$ does not necessarily hold true, since optimal stopping times are in general not unique. At some point $(t,x) \in [0,T]\times\mathbb{R}$ it might be optimal to stop immediately ($x \in D_t^\pi$) as well as to wait a positive amount of time until X hits A ($x \notin A_t$). A particularly simple example is the case where X is a martingale and $f(t,x) = 0$ and $g(t,x) = x$. The optional stopping theorem implies that all stopping times $\tau \in \mathcal{T}_{t,T}$ generate the same expected payoff $W(t,x,\tau) = x$. Therefore, every set A is implemented by the zero transfer. The stopping region consists of the whole state space $D_t^0 = \mathbb{R}$.

We introduce the notion of strict implementability, where we additionally require that outside of A it is not optimal to stop.

Definition 3 (Strict Implementability)**.** A time-closed set A is strictly implemented by a transfer π if A is implemented by π and $v^\pi(t,x) > g(t,x) + \pi(t)$ for all $x \notin A_t$ and $t \in [0,T]$.

In particular, every strictly implementable set A satisfies $A_t = D_t^\pi$ for the transfer π. Since the stopping regions D_t^π are closed (see Lemma 6) the restriction to time-closed sets is no loss of generality. Any set that is not time closed can not be strictly implemented.

Note that the notion of implementability generalizes the notion of optimal stopping times. If $\tau_A^{t,x}$ is an optimal stopping time in a stopping problem of the form

$$\sup_{\tau\in\mathcal{T}_{t,T}} \mathbb{E}\left[\int_t^\tau f(s,X_s^{t,x})\,\mathrm{d}s + g(\tau,X_\tau^{t,x})\right]$$

for all $(t,x)\in[0,T]\times\mathbb{R}$, then it is implemented by the zero transfer.

3.4. Single Crossing and Cutoff Regions

Next we introduce the main structural condition on the payoff functions.

Condition 4 (Single Crossing). We say that the single crossing condition is satisfied if for all $t\in[0,T]$ the mapping $x\mapsto f(t,x)+(\partial_t+\mathcal{L})g(t,x)$ is nonincreasing. If this monotonicity is strict, then we say that the strict single crossing condition holds.

Note that the (strict) single crossing condition is satisfied in a number of examples. For instance, it is satisfied in the examples of Sections 2.1 and 2.2.

Moreover, we define a special subclass of time-closed sets.[2]

Definition 5. A time-closed set A is called a cutoff region if there exists a function b: $[0,T]\to\bar{\mathbb{R}}$ such that $A_t=[b(t),\infty)$. In this case we call b the associated cutoff and we write

$$\tau_A^{t,x}=\tau_b^{t,x}=\inf\{s\geq t \mid X_s^{t,x}\geq b(s)\}\wedge T$$

for $(t,x)\in[0,T]\times\mathbb{R}$. We call τ_b a cutoff rule. We say that a cutoff region A is regular, if the associated cutoff b: $[0,T]\to\mathbb{R}$ is càdlàg (i.e., is right continuous and has left limits in $\mathbb{R}$) and has summable downward jumps, that is,

$$\sum_{0\leq s\leq t}(\Delta b_s)^- < \infty.$$

4. Strictly Implementable Regions Are Cutoff Regions

For optimal stopping problems it is well known that under the single crossing condition (or a weaker version of it) there exists a cutoff rule that is optimal (see, e.g., Kotlow [23], Jacka and Lynn [20], or Villeneuve [44]). In this section we show that the opposite direction holds more generally for strict implementability: only cutoff regions can be strictly implemented.

We first state the following regularity result about v^π.

Lemma 6. *For every transfer π and every $t\in[0,T]$ the mapping $x\mapsto v^\pi(t,x)$ is Lipschitz continuous. In particular, the stopping region D_t^π is closed.*

Proof. Fix $t\in[0,T]$ and $x,y\in\mathbb{R}$. By Lipschitz continuity of f and g there exists a constant $C>0$ such that

$$\begin{aligned}|v^\pi(t,x)-v^\pi(t,y)| &\leq \sup_{\tau\in\mathcal{T}_{t,T}}\mathbb{E}\left[\int_t^\tau |f(s,X_s^{t,x})-f(s,X_s^{t,y})|\,\mathrm{d}s + |g(\tau,X_\tau^{t,x})-g(\tau,X_\tau^{t,y})|\right]\\ &\leq C\mathbb{E}\left[\sup_{s\in[t,T]}|X_s^{t,x}-X_s^{t,y}|\right].\end{aligned}$$

By the well-known moment estimate for solutions of stochastic differential equations (see, e.g., Kunita [25, Theorem 3.2]) there exists a constant $\tilde{C}$ (independent of x and y) such that $\mathbb{E}[\sup_{s\in[t,T]}|X_s^{t,x}-X_s^{t,y}|]\leq\tilde{C}|x-y|$. This yields the claim. □

The next result shows that under the single-crossing condition only cutoff regions are strictly implementable.

Proposition 7. *Assume that the single crossing condition holds true. For every transfer π,*
(i) *the stopping region D_t^π is a cutoff region*
(ii) *and thus if A is strictly implemented by π then A is a cutoff region.*

Proof. Fix $t\in[0,T]$. First observe that the single crossing condition implies that $x\mapsto v^\pi(t,x)-g(t,x)$ is nonincreasing. Indeed, Itô's formula applied to $g(\cdot,X)$ yields

$$W(t,x,\tau)=\mathbb{E}\left[\int_t^\tau (f(s,X_s^{t,x})+(\partial_t+\mathscr{L})g(s,X_s^{t,x}))\,\mathrm{d}s+g(t,x)+\int_t^\tau g_x(s,X_s^{t,x})\sigma(s,X_s^{t,x})\,\mathrm{d}W_s\right]$$

for every $x\in\mathbb{R}$ and $\tau\in\mathscr{T}_{t,T}$. Since g_x is bounded and σ has linear growth, the process $\int_t^\cdot g_x(s,X_s^{t,x})\sigma(s,X_s^{t,x})\,\mathrm{d}W_s$ is a martingale. It follows from the comparison principle (7) and the single crossing condition that for $x\le y$

$$\begin{aligned}v^\pi(t,x)-g(t,x)&=\sup_{\tau\in\mathscr{T}_{t,T}}\mathbb{E}\left[\int_t^\tau (f(s,X_s^{t,x})+(\partial_t+\mathscr{L})g(s,X_s^{t,x}))\,\mathrm{d}s+\pi(\tau)\right]\\&\ge\sup_{\tau\in\mathscr{T}_{t,T}}\mathbb{E}\left[\int_t^\tau (f(s,X_s^{t,y})+(\partial_t+\mathscr{L})g(s,X_s^{t,y}))\,\mathrm{d}s+\pi(\tau)\right]\\&=v^\pi(t,y)-g(t,y).\end{aligned}$$

This implies that $y\in D_t^\pi$ if $y\ge x$ and $x\in D_t^\pi$. Hence D_t^π is an interval that is unbounded on the right. By Lemma 6 the set D_t^π is closed. Hence there exists some $b(t)\in\bar{\mathbb{R}}$ such that $D_t^\pi=[b(t),\infty)$. This implies that A is a cutoff region since $A_t=D_t^\pi$ by the definition of strict implementability. □

We note that if we do not restrict attention to transfers π that depend only on time, but allow for the transfer to depend on the value of the process X, then any measurable set A can be implemented. To see this, observe that when $\pi(t,x)=-g(x,t)+\mathbf{1}_{\{(x,t)\in A\}}$ the optimal stopping problem becomes

$$\sup_{\tau\in\mathscr{T}}\mathbb{E}[g(X_\tau,\tau)+\pi(X_\tau,\tau)]=\sup_{\tau\in\mathscr{T}}\mathbb{E}[\mathbf{1}_{\{(X_\tau,\tau)\in A\}}]$$

to which $\tau_A=\inf\{t\colon (X_t,t)\in A\}$ is a solution. By not allowing for spatial dependence in the transfer, the inverse problem becomes harder to solve. While the assumption of spatial independence makes our problem mathematically nontrivial it also has a clear economic motivation in dynamic principal agent applications in economics where the value of the process is privately observed by the agent and thus the transfer chosen by the principal can not condition on it.

5. Implementability of Cutoff Regions

In this section we prove that the converse implication of Proposition 7 holds true as well: Every regular cutoff region is implementable. We derive a closed form representation for the transfer in terms of the reflected version of X in Section 5.1. In Section 5.2 we verify that this candidate solution to the inverse optimal stopping problem indeed implements cutoff regions. The main properties of the transfer are presented in Section 5.3. In Section 5.4 we provide a uniqueness result for transfers implementing a cutoff region.

5.1. Reflected SDEs and a Formal Derivation of the Candidate Transfer

A solution to a reflected stochastic differential equation (RSDE) is a pair of processes $(\tilde{X},l)$, where the process $\tilde{X}$ evolves according to the dynamics of the associated SDE (6) below a given barrier b and is pushed below the barrier by the process l whenever it tries to exceed b. Next we give a formal definition.

Definition 8. Let b be a càdlàg barrier, $t\in[0,T]$ a fixed point in time and $\tilde{\xi}\le b(t)$ a $\mathscr{F}_t$-measurable square-integrable random variable. A pair $(\tilde{X},l)$ of adapted processes (with càdlàg trajectories) is called a (strong) solution to the stochastic differential equation (6) reflected at b with initial condition $(t,\tilde{\xi})$ if it satisfies the following properties:

(i) $\tilde{X}$ is constrained to stay below the barrier, that is, $\tilde{X}_s\le b(s)$ almost surely for every $s\in[t,T]$.

(ii) For every $s\in[t,T]$ the following integral equation holds almost surely

$$\tilde{X}_s=\tilde{\xi}+\int_t^s\mu(r,\tilde{X}_r)\,\mathrm{d}r+\int_t^s\sigma(r,\tilde{X}_r)\,\mathrm{d}W_r-l_s. \tag{9}$$

(iii) The process l starts in zero, is nondecreasing, and only increases when $\tilde{X}_t=b(t)$, that is,

$$\int_t^T (b(s)-\tilde{X}_s)\,\mathrm{d}l_s=0. \tag{10}$$

To stress the dependence of $\tilde{X}$ on the initial value we sometimes write $\tilde{X}^{t,\tilde{\xi}}$.

Remark 9. Consider the situation where b has a downward jump at time t and $\tilde{X}$ is above $b(t)$ shortly before time t, that is, $\tilde{X}_{t-}(\omega) \in (b(t), b(t-)]$ for some $\omega \in \Omega$. Since $\tilde{X}_t \le b(t)$ the reflected process $\tilde{X}$ has a downward jump at time t as well. Equation (9) implies that l has an upward jump at time t. Then Equation (10) yields that $\tilde{X}$ is on the barrier at time t, that is, $\tilde{X}_t = b(t)$. Hence, the jump of b is rather absorbed by $\tilde{X}$ than truly reflected (which would mean $\tilde{X}_t = 2b(t) - \tilde{X}_{t-}$). In this sense $\tilde{X}$ is the maximal version of X that stays below b. This property is crucial in the proof of Theorem 11. Existence and uniqueness of $\tilde{X}$ are established in Rutkowski [40]. We also refer to Slominski and Wojciechowski [43] who allow for general modes of reflection. For results about RSDEs with "true" jump reflections we refer to Chaleyat-Maurel et al. [6].

5.1.1. A Formal Derivation. Here we establish the link between inverse optimal stopping problems and RSDEs and derive the representation of a transfer implementing a cutoff region. To this end assume that the cutoff region $A = [b(t), \infty)$ is implemented by a transfer π. Without loss of generality we assume that $\pi(T) = 0$ (else take $\tilde{\pi}(t) = \pi(t) - \pi(T)$). Since we are only interested in a heuristic derivation here, we make some regularity assumptions. We assume that the value function of the stopping problem (8) is smooth ($v^\pi \in C^{1,2}([0,T] \times \mathbb{R})$) and that b is continuous such that $\tilde{X}$ is continuous as well. Then v^π satisfies (see, e.g., Peskir and Shiryaev [38, Chapter IV])

$$\min\{-(\partial_t + \mathscr{L})v^\pi - f, v^\pi - (g + \pi)\} = 0$$
$$v^\pi(T, \cdot) = g(T, \cdot)$$

and b is the free boundary of this variational partial differential equation. In particular, below the cutoff b the value function v^π satisfies the continuation equation

$$(\partial_t + \mathscr{L})v^\pi(t,x) = -f(t,x)$$

for all $x \le b(t)$. On the cutoff, v^π satisfies the boundary condition $v^\pi(t, b(t)) = g(t, b(t)) + \pi(t)$ for all $t \in [0,T]$. Moreover, if b is sufficiently regular the smooth fit principle

$$v_x(t, b(t)) = g_x(t, b(t))$$

holds for all $t \in [0,T]$ (see, e.g., Peskir and Shiryaev [38, Section 9.1]). Then Itô's formula implies

$$\begin{aligned}
\mathbb{E}[g(T, \tilde{X}_T^{t,b(t)})] &= \mathbb{E}[v^\pi(T, \tilde{X}_T^{t,b(t)})] \\
&= v^\pi(t, b(t)) + \mathbb{E}\left[\int_t^T (\partial_t + \mathscr{L})v^\pi(s, \tilde{X}_s^{t,b(t)})\,\mathrm{d}s - \int_t^T v_x(s, \tilde{X}_s^{t,b(t)})\,\mathrm{d}l_s\right] \\
&= g(t, b(t)) + \pi(t) - \mathbb{E}\left[\int_t^T f(s, \tilde{X}_s^{t,b(t)})\,\mathrm{d}s + \int_t^T g_x(s, \tilde{X}_s^{t,b(t)})\,\mathrm{d}l_s\right].
\end{aligned}$$

A further application of Itô's formula yields the following representation of π:

$$\begin{aligned}
\pi(t) &= \mathbb{E}\left[g(T, \tilde{X}_T^{t,b(t)}) + \int_t^T f(s, \tilde{X}_s^{t,b(t)})\,\mathrm{d}s + \int_t^T g_x(s, \tilde{X}_s^{t,b(t)})\,\mathrm{d}l_s\right] - g(t, b(t)) \\
&= \mathbb{E}\left[\int_t^T f(s, \tilde{X}_s^{t,b(t)}) + (\partial_t + \mathscr{L})g(s, \tilde{X}_s^{t,b(t)})\,\mathrm{d}s\right]. \qquad (11)
\end{aligned}$$

In Theorem 11 we verify that Equation (11) indeed leads to a transfer π implementing A. The proof does neither rely on any analytic methods nor on results from the theory of partial differential equations. Instead we employ purely probabilistic arguments based on the single crossing condition and comparison results for SDEs and RSDEs. This methodology requires weak regularity assumptions on the model parameters. In particular there is no ellipticity condition on σ.

5.1.2. Properties of RSDEs. The next proposition proves auxiliary results about RSDEs, which we will use in the proof of Theorem 11. There is a broad literature on RSDEs including comparison results (see, e.g., Bo and Yao [2]). To the best of our knowledge, the comparison principles for RSDE with càdlàg barriers and summable downward jumps as needed for our result have not been shown before. While all results follow by standard arguments we give a proof in the appendix for the convenience of the reader. For the existence and uniqueness result we refer to Rutkowski [40].

Proposition 10. *For every regular*[3] *cutoff b there exists a unique strong solution $\tilde{X}$ to the RSDE (9). The process l is given by*

$$l_s = \sup_{t\le r\le s}\left(\tilde{\xi} + \int_t^r \mu(u,\tilde{X}_u)\,\mathrm{d}u + \int_t^r \sigma(u,\tilde{X}_u)\,\mathrm{d}W_u - b(r)\right)^+. \tag{12}$$

Moreover, $\tilde{X}$ satisfies

(i) (*Square Integrability*) $\mathbb{E}[\sup_{t\le s\le T}(\tilde{X}_s^{t,\xi})^2] < \infty$ *for all* $t\in[0,T]$.
(ii) (*Minimality*) $\tilde{X}_s^{t,\xi}1_{\{s<\tau_b\}} = X_s^{t,\xi}1_{\{s<\tau_b\}}$ *a.s. for all* $s\in[t,T]$.
(iii) (*Comparison Principle for the Reflected Process*) *If* $\xi_1\le\xi_2$ *a.s., then for* $s\in[t,T]$ *we have* $\tilde{X}_s^{t,\xi_1}\le\tilde{X}_s^{t,\xi_2}$ *a.s.*
(iv) (*Moment Estimate*) *There exists a constant* $K>0$ *such that* $\mathbb{E}[\sup_{t\le r\le s}|\tilde{X}_r^{t,\xi_1}-\tilde{X}_r^{t,\xi_2}|^p \mid \mathcal{F}_t]\le K|\xi_1-\xi_2|^p$ *a.s. for all* $t\in[0,T]$, $s\in[t,T]$, $\xi_1,\xi_2\in L^2(\mathcal{F}_t)$ *and* $p=1,2$.
(v) (*Comparison Principle for the Original Process*) $\tilde{X}_s^{t,\xi}\le X_s^{t,\xi}$ *a.s. for all* $s\in[t,T]$.
(vi) (*Left continuity*) *Let* $t\in[0,T]$ *and* $x\le b(t)\wedge b(t-)$. *Then* $\tilde{X}_t^{s,y\wedge b(s)}\to x$ *in* L^2 *for* $s\nearrow t$ *and* $y\to x$.

Using similar arguments as in Protter [39, Chapter V Section 6] one can show that $\tilde{X}$ satisfies the strong Markov property. For $s\ge t$ we define the transition kernel $\tilde{P}_{t,s}$ by

$$\tilde{P}_{t,s}\varphi(t,x) = \mathbb{E}[\varphi(s,\tilde{X}_s^{t,x})]$$

for any Borel measurable, bounded function $\varphi\colon [0,T]\times\mathbb{R}\to\mathbb{R}$. Then $\tilde{X}$ satisfies for any stopping time $\tau\in\mathcal{T}$ and $u\ge 0$

$$\mathbb{E}[\varphi(\tau+u,\tilde{X}_{\tau+u})\mid\mathcal{F}_\tau] = \tilde{P}_{\tau,\tau+u}\varphi(\tau,\tilde{X}_\tau). \tag{13}$$

Moreover, uniqueness of strong solutions of RSDEs implies the following flow property of $\tilde{X}$. For $t\le r\le s$ and $x\in\mathbb{R}$ we have a.s.

$$\tilde{X}_s^{t,x} = \tilde{X}_s^{r,\tilde{X}_r^{t,x}}. \tag{14}$$

5.2. Regular Cutoff Regions Are Implementable

In this section we prove our main theorem stating that every regular cutoff region is implemented by the transfer derived in Section 5.1.

Theorem 11. *Assume that the single crossing condition is satisfied. Let A be a regular cutoff region with boundary b. Then it is implemented by the transfer*

$$\pi(t) = \mathbb{E}\left[\int_t^T f(s,\tilde{X}_s^{t,b(t)}) + (\partial_t+\mathcal{L})g(s,\tilde{X}_s^{t,b(t)})\,\mathrm{d}s\right]. \tag{15}$$

Proof. First observe that the cutoff rule $\tau_b^{t,x}$ is a stopping time for all $(t,x)\in[0,T]\times\mathbb{R}$. Indeed, since X has continuous paths and b is right continuous, the Début-theorem (see, e.g., Dellacherie and Meyer [8, Chapter IV, Section 50]) implies $\tau_b^{t,x}\in\mathcal{T}_{t,T}$.

Let π be given by Equation (15). For the boundedness and measurability of π we refer to Proposition 13. We set $h = f + (\partial_t+\mathcal{L})g$. As in the proof of Proposition 7 we have

$$W(t,x,\tau) = g(t,x) + \mathbb{E}\left[\int_t^\tau h(s,X_s^{t,x})\,\mathrm{d}s\right].$$

Note that we can write π in terms of the transition function $\tilde{P}$ of $\tilde{X}$ as follows

$$\pi(t) = \int_t^T \tilde{P}_{t,s}h(t,b(t))\,\mathrm{d}s.$$

The strong Markov property (Equation (13)) of $\tilde{X}$ implies

$$\tilde{P}_{\tau,\tau+u}h(\tau,b(\tau)) = \mathbb{E}[h(\tau+u,\tilde{X}_{\tau+u}^{\tau,b(\tau)})\mid\mathcal{F}_\tau]$$

for any stopping time $\tau\in\mathcal{T}$ and $u\ge 0$. Hence we have

$$\pi(\tau) = \mathbb{E}\left[\int_\tau^T h(s,\tilde{X}_s^{\tau,b(\tau)})\,\mathrm{d}s\,\middle|\,\mathcal{F}_\tau\right]. \tag{16}$$

Fix $t \in [0,T]$ and $x \geq b(t)$. Let $\tau \in \mathcal{T}_{t,T}$ be an arbitrary stopping time. The comparison principle between the original and the reflected process (Property (v)) implies $X_s^{t,x} \geq X_s^{t,b(t)} \geq \tilde{X}_s^{t,b(t)}$ a.s. for every $s \in [t,T]$. From the flow property (Equation (14)) and the comparison principle for reflected processes (Property (iii)) follows that $\tilde{X}_s^{t,b(t)} = \tilde{X}_s^{\tau,\tilde{X}_\tau^{t,b(t)}} \leq \tilde{X}_s^{\tau,b(\tau)}$ a.s. for every $s \in [\tau,T]$. Therefore the single crossing condition implies

$$\begin{aligned}\mathbb{E}\bigg[\int_t^\tau h(s,X_s^{t,x})\,\mathrm{d}s + \pi(\tau)\bigg] &= \mathbb{E}\bigg[\int_t^\tau h(s,X_s^{t,x})\,\mathrm{d}s + \int_\tau^T h(s,\tilde{X}_s^{\tau,b(\tau)})\,\mathrm{d}s\bigg]\\ &\leq \mathbb{E}\bigg[\int_t^\tau h(s,\tilde{X}_s^{t,b(t)})\,\mathrm{d}s + \int_\tau^T h(s,\tilde{X}_s^{t,b(t)})\,\mathrm{d}s\bigg]\\ &= \pi(t).\end{aligned}$$

This implies $W(t,x,\tau) + \mathbb{E}[\pi(\tau)] \leq W(t,x,t) + \pi(t)$. Hence $\tau_b^{t,x} = t$ is optimal in (8) as claimed.

In the second step, fix $x < b(t)$ and let $\tau \in \mathcal{T}_{t,T}$ be an arbitrary stopping time. To shorten notation we write $\tau_b = \tau_b^{t,x}$. First, we prove that the stopping $\min\{\tau,\tau_b\}$ performs at least as well as τ. By (16) we have

$$\mathbb{E}[1_{\{\tau_b<\tau\}}\pi(\tau)] = \mathbb{E}\bigg[1_{\{\tau_b<\tau\}}\mathbb{E}\bigg[\int_\tau^T h(s,\tilde{X}_s^{\tau,b(\tau)})\,\mathrm{d}s \,\bigg|\, \mathcal{F}_\tau\bigg]\bigg] = \mathbb{E}\bigg[1_{\{\tau_b<\tau\}}\int_\tau^T h(s,\tilde{X}_s^{\tau,b(\tau)})\,\mathrm{d}s\bigg].$$

This leads to

$$\mathbb{E}\bigg[1_{\{\tau_b<\tau\}}\bigg(\int_t^\tau h(s,X_s^{t,x})\,\mathrm{d}s + \pi(\tau)\bigg)\bigg] = \mathbb{E}\bigg[1_{\{\tau_b<\tau\}}\bigg(\int_t^{\tau_b} h(s,X_s^{t,x})\,\mathrm{d}s + \int_{\tau_b}^\tau h(s,X_s^{t,x})\,\mathrm{d}s + \int_\tau^T h(s,\tilde{X}_s^{\tau,b(\tau)})\,\mathrm{d}s\bigg)\bigg].$$

By construction of the reflected process $\tilde{X}$ we have $\tilde{X}_{\tau_b}^{t,x} = b(\tau_b)$. The comparison principle between the original and the reflected process (Property (v)) and the flow property of reflected processes (Equation (14)) imply almost surely

$$\tilde{X}_s^{\tau_b,b(\tau_b)} = \tilde{X}_s^{\tau_b,\tilde{X}_{\tau_b}^{t,x}} = \tilde{X}_s^{t,x} \leq X_s^{t,x}$$

for $s \geq \tau_b$. Since $\tilde{X}_\tau^{\tau_b,b(\tau_b)} \leq b(\tau)$ we have on the set $\{\tau > \tau_b\}$

$$\tilde{X}_s^{\tau,b(\tau)} \geq \tilde{X}_s^{\tau,\tilde{X}_\tau^{\tau_b,b(\tau_b)}} = \tilde{X}_s^{\tau_b,b(\tau_b)}$$

for all $s \geq \tau$. These two inequalities combined with the monotonicity of h yield that

$$\begin{aligned}&\mathbb{E}\bigg[1_{\{\tau_b<\tau\}}\bigg(\int_t^\tau h(s,X_s^{t,x})\,\mathrm{d}s + \pi(\tau)\bigg)\bigg]\\ &\quad\leq \mathbb{E}\bigg[1_{\{\tau_b<\tau\}}\bigg(\int_t^{\tau_b} h(s,X_s^{t,x})\,\mathrm{d}s + \int_{\tau_b}^\tau h(s,\tilde{X}_s^{\tau_b,b(\tau_b)})\,\mathrm{d}s + \int_\tau^T h(s,\tilde{X}_s^{\tau_b,b(\tau_b)})\,\mathrm{d}s\bigg)\bigg]\\ &\quad= \mathbb{E}\bigg[1_{\{\tau_b<\tau\}}\bigg(\int_t^{\tau_b} h(s,X_s^{t,x})\,\mathrm{d}s + \pi(\tau_b)\bigg)\bigg].\end{aligned}$$

Consequently, using the stopping time $\min\{\tau,\tau_b\}$ is at least as good as using τ

$$\begin{aligned}W(t,x,\tau) + \mathbb{E}[\pi(\tau)] &= g(t,x) + \mathbb{E}\bigg[\int_t^\tau h(s,X_s^{t,x})\,\mathrm{d}s + \pi(\tau)\bigg]\\ &\leq g(t,x) + \mathbb{E}\bigg[\int_t^{\tau\wedge\tau_b} h(s,X_s^{t,x})\,\mathrm{d}s + \pi(\min\{\tau,\tau_b\})\bigg]\\ &= W(t,x,\min\{\tau,\tau_b\}) + \mathbb{E}[\pi(\min\{\tau,\tau_b\})].\end{aligned}$$

Thus it suffices to consider stopping rules $\tau \leq \tau_b$. In this case we have

$$\mathbb{E}\bigg[\int_t^\tau h(s,X_s^{t,x})\,\mathrm{d}s + \pi(\tau)\bigg] = \mathbb{E}\bigg[\int_t^\tau h(s,X_s^{t,x})\,\mathrm{d}s + \int_\tau^{\tau_b} h(s,\tilde{X}_s^{\tau,b(\tau)})\,\mathrm{d}s + \int_{\tau_b}^T h(s,\tilde{X}_s^{\tau,b(\tau)})\,\mathrm{d}s\bigg].$$

From the comparison principle for reflected processes (Property (iii)) and the flow property Equation (14) follows $\tilde{X}_s^{t,x} = \tilde{X}_s^{\tau,\tilde{X}_\tau^{t,x}} \leq \tilde{X}_s^{\tau,b(\tau)}$ for all $s \geq \tau$. By the minimality property of reflected processes (Property (ii)) we have that $X_s^{t,x} = \tilde{X}_s^{t,x}$ for all $s < \tau_b$. Similar considerations as above yield

$$\tilde{X}_s^{\tau_b,b(\tau_b)} = \tilde{X}_s^{\tau_b,\tilde{X}_{\tau_b}^{t,x}} = \tilde{X}_s^{t,x} = \tilde{X}_s^{\tau,\tilde{X}_\tau^{t,x}} \leq \tilde{X}_s^{\tau,b(\tau)}$$

a.s. for $s \geq \tau_b$. The monotonicity of h implies

$$\begin{aligned}\mathbb{E}\left[\int_t^\tau h(s,X_s^{t,x})\,\mathrm{d}s + \pi(\tau)\right] &\leq \mathbb{E}\left[\int_t^\tau h(s,X_s^{t,x})\,\mathrm{d}s + \int_\tau^{\tau_b} h(s,X_s^{t,x})\,\mathrm{d}s + \int_{\tau_b}^T h(s,\tilde{X}_s^{\tau_b,b(\tau_b)})\,\mathrm{d}s\right]\\ &= \mathbb{E}\left[\int_t^{\tau_b} h(s,X_s^{t,x})\,\mathrm{d}s + \pi(\tau_b)\right]\end{aligned}$$

and hence $W(t,x,\tau) + \mathbb{E}[\pi(\tau)] \leq W(t,x,\tau_b) + \mathbb{E}[\pi(\tau_b)]$. This completes the proof of implementability. □

Theorem 11 shows that every cutoff stopping time is implementable under the single crossing condition we imposed. We note that this result does not hold without the single crossing condition. To see this, consider as an example a payoff $g(x,t) = h(|x|,t)$, which is only a function of the absolute value of x and a symmetric diffusion process $\mu(x,t) = -\mu(-x,t)$ and $\sigma(x,t) = \sigma(-x,t)$. Note, that such an example never satisfies the single crossing condition. As for any $\pi\colon \mathbb{R}_+ \to \mathbb{R}$ the optimal stopping problem

$$\sup_{\tau\in\mathcal{T}} \mathbb{E}[g(X_\tau,\tau) + \pi(\tau)]$$

is symmetric at zero, it follows that the stopping set must be symmetric around zero. Consequently, the agent does not only stop when the process X crosses a threshold from below, but also when X crosses the negative of this threshold from above. Hence, the optimal stopping time is never of cutoff form, and no cutoff rule can be implemented.

In Proposition 7 we showed that strictly implementable regions are necessarily of cutoff type. The next result establishes the converse direction. Under the strict single crossing condition, cutoff regions are strictly implementable.

Corollary 12. *If the strict single crossing condition holds true, then a regular cutoff region with barrier b is strictly implemented by the transfer from Equation* (15).

Proof. We use the same notation as in the proof of Theorem 11. Let $t \in [0,T]$ and $x < b(t)$. Then the right continuity of b and $\tilde{X}$ and the strict monotonicity of h imply that

$$\mathbb{E}\left[\int_t^{\tau_b} h(s,\tilde{X}_s^{t,x})\,\mathrm{d}s\right] > \mathbb{E}\left[\int_t^{\tau_b} h(s,\tilde{X}_s^{t,b(t)})\,\mathrm{d}s\right].$$

Consequently we have

$$\begin{aligned}\mathbb{E}\left[\int_t^{\tau_b} h(s,X_s^{t,x})\,\mathrm{d}s + \pi(\tau_b)\right] &= \mathbb{E}\left[\int_t^{\tau_b} h(s,\tilde{X}_s^{t,x})\,\mathrm{d}s + \int_{\tau_b}^T h(s,\tilde{X}_s^{\tau_b,b(\tau_b)})\,\mathrm{d}s\right]\\ &> \mathbb{E}\left[\int_t^{\tau_b} h(s,\tilde{X}_s^{t,b(t)})\,\mathrm{d}s + \int_{\tau_b}^T h(s,\tilde{X}_s^{t,b(t)})\,\mathrm{d}s\right]\\ &= \pi(t).\end{aligned}$$

This implies $v^\pi(t,x) > \pi(t) + g(t,x)$ and hence A is strictly implemented by π. □

In general, the distribution of the reflected process $\tilde{X}$ is not explicitly known. Hence, one has to fall back to numerical methods to approximate the transfer from Theorem 11. For example, one could use discretization schemes for the RSDE (9) and Monte Carlo simulations to evaluate the expectation in Equation (15) (see, e.g., Saisho [41], Bossy et al. [4] or Önskog and Nyström [30]).

5.3. Properties of the Transfer

The next proposition summarizes properties of transfer implementing a cutoff region.

Proposition 13. *Let b: $[0,T]\to\mathbb{R}$ be a regular cutoff. The transfer π from Equation (15) satisfies the following properties:*

(i) *π is càdlàg. In particular π is bounded and measurable.*
(ii) *π is continuous at $t\in[0,T]$ if b is continuous at t or if b has a downward jump at t.*
(iii) *π has no upward jumps.*
(iv) *If π has a downward jump at $t\in[0,T]$, then b has an upward jump at t.*
(iv) *π converges to 0 at time T: $\lim_{t\nearrow T}\pi(t)=0$.*

Proof. As in the proof of Theorem 11 we introduce the function $h(t,x)=f(t,x)+(\partial_t+\mathscr{L})g(t,x)$. By assumption h is Lipschitz continuous and has linear growth in x. The transfer π is given by

$$\pi(t)=\mathbb{E}\left[\int_t^T h(s,\tilde{X}_s^{t,b(t)})\,\mathrm{d}s\right].$$

We first show that π is right continuous. For $t\in[0,T]$ and $\epsilon>0$ we have

$$|\pi(t)-\pi(t+\epsilon)|\le\mathbb{E}\left[\int_t^{t+\epsilon}|h(s,\tilde{X}_s^{t,b(t)})|\,\mathrm{d}s\right]+\mathbb{E}\left[\int_{t+\epsilon}^T|h(s,\tilde{X}_s^{t,b(t)})-h(s,\tilde{X}_s^{t+\epsilon,b(t+\epsilon)})|\,\mathrm{d}s\right].$$

It follows from the linear growth of h and Property (i) of $\tilde{X}$ from Proposition 10 that $\mathbb{E}[\int_t^{t+\epsilon}|h(s,\tilde{X}_s^{t,b(t)})|\,\mathrm{d}s]\to 0$ as $\epsilon\to 0$. Moreover, the Lipschitz continuity of h implies

$$\mathbb{E}\left[\int_{t+\epsilon}^T|h(s,\tilde{X}_s^{t,b(t)})-h(s,\tilde{X}_s^{t+\epsilon,b(t+\epsilon)})|\,\mathrm{d}s\right]\le C\mathbb{E}\left[\sup_{s\in[t+\epsilon,T]}|\tilde{X}_s^{t,b(t)}-\tilde{X}_s^{t+\epsilon,b(t+\epsilon)}|\right]$$

for some constant $C>0$. By the flow property (Equation (14)) we have $\tilde{X}_s^{t,b(t)}=\tilde{X}_s^{t+\epsilon,\tilde{X}_{t+\epsilon}^{t,b(t)}}$. Property (iv) from Proposition 10 yields

$$\mathbb{E}\left[\sup_{s\in[t+\epsilon,T]}|\tilde{X}_s^{t,b(t)}-\tilde{X}_s^{t+\epsilon,b(t+\epsilon)}|\right]\le\tilde{C}\mathbb{E}[|\tilde{X}_{t+\epsilon}^{t,b(t)}-b(t+\epsilon)|].$$

Right continuity of $\tilde{X}$ and b then implies $\pi(t+)=\pi(t)$.[4]

Concerning the left-hand limits of π we show that

$$\pi(t-)=\mathbb{E}\left[\int_t^T h(s,\tilde{X}_s^{t,b(t)\wedge b(t-)})\,\mathrm{d}s\right]\tag{17}$$

for all $t\in(0,T]$. Equation (17) implies all remaining claims of Proposition 13. If b is continuous at t or has a downward jump ($b(t)\le b(t-)$), then Equation (17) yields continuity of π at t: $\pi(t-)=\pi(t)$. Monotonicity of h and the comparison principle for the reflected process imply $\pi(t-)\ge\pi(t)$, that is, π has no upward jumps. If π has a downward jump at time t ($\pi(t-)>\pi(t)$), then Equation (17) yields that b has necessarily an upward jump ($b(t)>b(t-)$). Moreover, it follows from Equation (17) that $\pi(T-)=0$. To prove Equation (17) let $t\in(0,T]$ and $\epsilon>0$. Then consider

$$\left|\pi(t-\epsilon)-\mathbb{E}\left[\int_t^T h(s,\tilde{X}_s^{t,b(t)\wedge b(t-)})\,\mathrm{d}s\right]\right|\le\mathbb{E}\left[\int_{t-\epsilon}^t|h(s,\tilde{X}_s^{t-\epsilon,b(t-\epsilon)})|\,\mathrm{d}s\right]+\mathbb{E}\left[\int_t^T|h(s,\tilde{X}_s^{t-\epsilon,b(t-\epsilon)})-h(s,\tilde{X}_s^{t,b(t)\wedge b(t-)})|\,\mathrm{d}s\right].$$

By Property (vi) from Proposition 10 we have $\tilde{X}_s^{t-\epsilon,b(t-\epsilon)}\to\tilde{X}_s^{t,b(t)\wedge b(t-)}$ in L^2 as $\epsilon\searrow 0$. Lipschitz continuity and linear growth of h then imply that $\mathbb{E}[\int_{t-\epsilon}^t|h(s,\tilde{X}_s^{t-\epsilon,b(t-\epsilon)})|\,\mathrm{d}s]\to 0$ and $\mathbb{E}[\int_t^T|h(s,\tilde{X}_s^{t-\epsilon,b(t-\epsilon)})-h(s,\tilde{X}_s^{t,b(t)\wedge b(t-)})|\,\mathrm{d}s]\to 0$ for $\epsilon\searrow 0$. This yields the claim. □

5.4. Uniqueness of the Transfer

To prove a uniqueness result for the transfer from Theorem 11 we need the following auxiliary result about cutoff stopping times.

Lemma 14. *Let b: $[0,T]\to\mathbb{R}$ be bounded from below. Then we have $\tau_b^{t,x}\nearrow T$ a.s. for $x\searrow-\infty$ and for every $t\in[0,T]$.*

Proof. Fix $t \in [0,T]$. By Kunita [25, Lemma 3.7] there exists a constant $C > 0$ such that

$$\mathbb{E}\left[\sup_{t\le s\le T}\left(\frac{1}{1+(X_s^{t,x})^2}\right)^2\right] \le C\left(\frac{1}{1+x^2}\right)^2.$$

Then Fatou's Lemma implies

$$\mathbb{E}\left[\liminf_{x\to-\infty}\sup_{t\le s\le T}\left(\frac{1}{1+(X_s^{t,x})^2}\right)^2\right] \le \liminf_{x\to-\infty}\mathbb{E}\left[\sup_{t\le s\le T}\left(\frac{1}{1+(X_s^{t,x})^2}\right)^2\right] \le \liminf_{x\to-\infty} C\left(\frac{1}{1+x^2}\right)^2 = 0.$$

Consequently we have $\limsup_{x\to-\infty}\inf_{t\le s\le T}|X_s^{t,x}| = \infty$ a.s. Together with the comparison principle for X this yields $\limsup_{x\to-\infty}\sup_{t\le s\le T} X_s^{t,x} = -\infty$ a.s. It follows that $\tau_b^{t,x}\nearrow T$ for $x\searrow -\infty$. □

Theorem 15. *Let A be a regular cutoff region with boundary b. Assume that A is implemented by two transfers π and $\hat\pi$ satisfying $\lim_{t\nearrow T}\pi(t) = \lim_{t\nearrow T}\hat\pi(t)$. Then $\pi(t) = \hat\pi(t)$ for all $t\in[0,T)$.*

Proof. Fix $t \in [0,T)$. To shorten notation we set $v = v^\pi$ and $\hat v = v^{\hat\pi}$. By Lemma 6 the functions v and $\hat v$ are Lipschitz continuous in the x variable. Similar considerations yield that the function $x\mapsto W(t,x,\tau)$ is Lipschitz continuous for every $\tau\in\mathcal{T}_{t,T}$. In particular, these functions are absolutely continuous. Appealing to the envelope theorem from Milgrom and Segal [28, Theorem 1] yields that

$$v_x(t,x) = W_x(t,x,\tau_b^{t,x}) = \hat v_x(t,x)$$

for Lebesgue almost every $x\in\mathbb{R}$. Integrating from $x < b(t)$ to $b(t)$ gives

$$v(t,b(t)) - v(t,x) = \hat v(t,b(t)) - \hat v(t,x)$$

or equivalently

$$\pi(t) - \hat\pi(t) = \mathbb{E}[\pi(\tau_b^{t,x}) - \hat\pi(\tau_b^{t,x})].$$

Since π and $\hat\pi$ are bounded we can appeal to Lemma 14 to obtain

$$\pi(t) - \hat\pi(t) = \lim_{x\to-\infty}\mathbb{E}[\pi(\tau_b^{t,x}) - \hat\pi(\tau_b^{t,x})] = 0,$$

where we used the dominated convergence theorem. □

6. Application to Optimal Stopping

From Theorem 15 we derive a probabilistic characterization of optimal stopping times for stopping problems of the form

$$v(t,x) = \sup_{\tau\in\mathcal{T}_{t,T}}\mathbb{E}\left[\int_t^\tau f(s,X_s^{t,x})\,ds + g(\tau,X_\tau^{t,x})\right], \tag{18}$$

where f, g, and X satisfy the single crossing condition. We say that a stopping time $\tau\in\mathcal{T}_{t,T}$ is optimal in (18) for $(t,x)\in[0,T]\times\mathbb{R}$ if

$$v(t,x) = \mathbb{E}\left[\int_t^\tau f(s,X_s^{t,x})\,ds + g(\tau,X_\tau^{t,x})\right].$$

Corollary 16. *Assume that the single crossing condition is satisfied and let $b\colon[0,T]\to\mathbb{R}$ be a regular cutoff. The stopping time $\tau_b^{t,x}$ is optimal in (18) for all $(t,x)\in[0,T]\times\mathbb{R}$, if and only if b satisfies the nonlinear integral equation*

$$\mathbb{E}\left[\int_t^T f(s,\tilde X_s^{t,b(t)}) + (\partial_t + \mathcal{L})g(s,\tilde X_s^{t,b(t)})\,ds\right] = 0 \tag{19}$$

for all $t\in[0,T]$.

Proof. First assume that (19) holds true for every $t\in[0,T]$. Then Theorem 11 implies that the cutoff region with boundary b is implemented by the zero transfer. This means that $\tau_b^{t,x}$ is optimal in (18) for every $(t,x)\in[0,T]\times\mathbb{R}$.

For the converse direction assume that $\tau_b^{t,x}$ is optimal in (18) for every $(t,x)\in[0,T]\times\mathbb{R}$. Then the cutoff region with boundary b is implemented by the zero transfer $\hat\pi = 0$. By Theorem 11 it is also implemented by the transfer $\pi(t) = \mathbb{E}[\int_t^T f(s,\tilde X_s^{t,b(t)}) + (\partial_t+\mathcal{L})g(s,\tilde X_s^{t,b(t)})\,ds]$. By Proposition 13 the transfer π satisfies $\lim_{t\nearrow T}\pi(t) = 0$. Then Theorem 15 implies that $\pi(t) = \hat\pi(t) = 0$ for all $t\in[0,T]$. □

In the literature on optimal stopping there is a well-known link between optimal stopping boundaries and a nonlinear integral equation differing from Equation (19). It was independently derived by Kim [22], Jacka [19], and Carr et al. [5] who considered the optimal exercise of an American option. Using the *early exercise premium representation* of the price of an American option, the authors arrive at a nonlinear integral equation that is satisfied by the optimal exercise boundary. The question whether the optimal exercise boundary is the only solution to the integral equation was left open, until more than a decade later (Peskir [32]) answered it in the affirmative. Using the change-of-variable formula with local time on curves derived in Peskir [31], allows Peskir [32] to characterize the optimal exercise boundary as the unique solution of the nonlinear integral equation in the class of continuous functions. The methodology of Peskir [32] was subsequently applied to solve optimal stopping problems with more general diffusion and Markov processes, multiple stopping boundaries, and more general payoff functionals. These problems include, for example, the optimal exercise of Russian (Peskir [33]) and British options (Peskir and Samee [36, 37]), the Wiener disorder problem (Gapeev and Peskir [14]), sequential testing problems (Gapeev and Peskir [13], Zhitlukhin and Muravlev [45]), the optimal stopping problem for maxima in diffusion models (Gapeev et al. [16]), optimal prediction problems (Du Toit and Peskir [10]), Bayesian disorder problems (Zhitlukhin and Shiryaev [46]), optimal liquidation problems (Ekstroem and Vaicenavicius [11]), and multiple optimal stopping problems (De Angelis and Kitapbayev [7]). In the framework of the present paper this integral equation is given by

$$\mathbb{E}\left[\int_t^T (f(s,X_s^{t,b(t)}) + (\partial_t + \mathscr{L})g(s,X_s^{t,b(t)}))\mathbf{1}_{\{X_s^{t,b(t)}\le b(s)\}}\,\mathrm{d}s\right] = 0 \tag{20}$$

(cf. Peskir and Shiryaev [38, Chapter IV, Section 14]).

Besides its interpretation in terms of the early exercise premium, Equation (20) is valuable from a numerical point of view. Indeed, it only requires for every $s\in[0,T]$ the law of X_s, which, when not known explicitly, can be approximated in various ways (e.g., using the Kolmogorov forward equation or Euler-Maruyama schemes). Once these distributions are available, (20) is a nonlinear Volterra (or Fredholm) integral equation that can be tackled using well-established numerical schemes provided in the literature. We also refer to the work of Belomestny and Gapeev [1], where an iterative procedure is proposed to approximate the solution of the integral equation and the value function. In contrast, it is not clear whether (19) can be numerically solved with high accuracy, since it is path dependent in terms of b. In particular, it is not possible to compute the marginal laws of $\tilde{X}$ upfront, as the unknown boundary b is entangled in the process $\tilde{X}$ (the distribution of the random variable $\tilde{X}_s^{t,b(t)}$ depends on the whole barrier $(b(r))_{t\le r\le s}$ from time t to s).

We also mention that the change of variables formula of Peskir [31], was extended in Peskir [34] to the multidimensional setting. This allows to characterize optimal stopping times as first hitting times also in higher dimensions (see, e.g., Glover et al. [17], Gapeev and Shiryaev [15], and Peskir [35]). Whether an extension of the methodology presented here to the multidimensional setting is possible is not clear. Already the formulation of the monotonicity in the single crossing condition (Condition 4) in higher dimensions is not straightforward. The construction of multivariate reflected processes is also highly nontrivial.

In general, the set of solutions to (19) is included in the set of solutions to (20). Indeed, if b solves (19) then by Corollary 16 it is an optimal stopping boundary and thus, under appropriate regularity conditions, it is also a solution to (20). In the cases where uniqueness holds for (20), the converse implication holds true as well. In the case of a constant barrier $b(t)=b\in\mathbb{R}$ and X a Brownian motion one can directly relate the two equations. Indeed, in this case it follows from the reflection principle that for all $x\le b$

$$\mathbb{P}[\tilde{X}_s^{t,b(t)}\le x] = 2\,\mathbb{P}[X_s^{t,b(t)}\le x]$$

and thus the constant barrier b solves Equation (20) if and only if it solves Equation (19). The question whether one can, in general, relate the two integral equations without taking the detour via optimal stopping problems is left open for future research.

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Appendix A

Proof of Proposition 10. Existence and uniqueness of $(\tilde{X}, l)$ follow from Rutkowski [40]. See also Slominski and Wojciechowski [43, Theorem 3.4] for the time-homogeneous case. By construction of $(\tilde{X}, l)$ we also have (i).

We next show (ii). Note that the solution to the unreflected SDE (6) solves the reflected SDE for $s < \tau_b$. As the solution to the reflected SDE is unique, (ii) follows.

To prove (iii) and (iv) we consider without loss of generality only the case $t = 0$. For $\xi_1, \xi_2 \in \mathbb{R}$ we write $(\tilde{X}^i, l^i) = (\tilde{X}^{0,\xi_i}, l^{0,\xi_i})$, $(i = 1, 2)$ and introduce the processes $D_t = \tilde{X}^1_t - \tilde{X}^2_t$ and $\Gamma_t = \sup_{s \le t} \max(0, D_s)^2$. Applying the Meyer-Itô formula Protter [39, Theorem 71, Chapter 4] to the function $x \mapsto \max(0, x)^2$ yields

$$\max(0, D_s)^2 = \max(0, D_0)^2 + 2\int_0^s 1_{\{D_{r-} > 0\}} D_{r-}\, \mathrm{d}D_r + \int_0^s 1_{\{D_{r-} > 0\}}\, \mathrm{d}[D]^c_r + \sum_{0 < r \le s} (\max(0, D_r)^2 - \max(0, D_{r-})^2 - 1_{\{D_{r-} > 0\}} D_{r-} \Delta D_r). \tag{A.1}$$

Since D only jumps when b has a downward jump and since $\tilde{X}^i$ jumps to the barrier, we have $-(\Delta b(r))^- \le \Delta D_r \le 0$ on the set $\{D_{r-} > 0\}$. Moreover, D has bounded paths. Since b has summable downward jumps we have $\sum_{0 < r \le s} 1_{\{D_{r-} > 0\}} |D_{r-} \Delta D_r| < \infty$ a.s. Hence, we can rewrite Equation (A.1) as follows:

$$\max(0, D_s)^2 = \max(0, D_0)^2 + 2\int_0^s 1_{\{D_r > 0\}} D_r\, \mathrm{d}D^c_r + \int_0^s 1_{\{D_{r-} > 0\}}\, \mathrm{d}[D]^c_r + \sum_{0 < r \le s} (\max(0, D_r)^2 - \max(0, D_{r-})^2). \tag{A.2}$$

Regarding the jump terms in Equation (A.2), assume that there exists $r \in (0, s]$ such that $\max(0, D_r)^2 > \max(0, D_{r-})^2$. This implies $D_r > 0$ and $D_r > D_{r-}$. Since $\tilde{X}^i$ jumps if and only if l^i jumps $(i = 1, 2)$ we obtain $\tilde{X}^1_r > \tilde{X}^2_r$ and $l^2_r - l^2_{r-} > l^1_r - l^1_{r-}$. It follows that $l^2_r - l^2_{r-} > 0$, since l^1 is nondecreasing. Hence, l^2 jumps at r, which implies that $\tilde{X}^2_r = b(r)$. Thus, we obtain the contradiction $\tilde{X}^1_r > b(r)$. Therefore we have

$$\sum_{0 < r \le s} (\max(0, D_r)^2 - \max(0, D_{r-})^2) \le 0.$$

For the last integral in Equation (A.2) the Lipschitz continuity of σ implies

$$\int_0^s 1_{\{D_r > 0\}}\, \mathrm{d}[D]^c_r = \int_0^s 1_{\{D_r > 0\}} (\sigma(r, \tilde{X}^1_r) - \sigma(r, \tilde{X}^2_r))^2\, \mathrm{d}r \le L^2 \int_0^s \max(0, D_r)^2\, \mathrm{d}r \le L^2 \int_0^s \Gamma_r\, \mathrm{d}r.$$

The first integral of Equation (A.2) decomposes into the following terms, which we will consider successively. By the Lipschitz continuity of μ we have

$$2\int_0^s 1_{\{D_r > 0\}} D_r (\mu(r, \tilde{X}^1_r) - \mu(r, \tilde{X}^2_r))\, \mathrm{d}r \le 2L \int_0^s \max(0, D_r)^2\, \mathrm{d}r \le L \int_0^s \Gamma_r\, \mathrm{d}r.$$

Next, we have

$$-2\int_0^s 1_{\{D_r > 0\}} D_r\, \mathrm{d}l^{1,c}_r \le 0$$

and

$$2\int_0^s 1_{\{D_r > 0\}} D_r\, \mathrm{d}l^{2,c}_r = 2\int_0^s 1_{\{\tilde{X}^1_r > b(r)\}} D_r\, \mathrm{d}l^{2,c}_r = 0.$$

Moreover, it follows from the Burkholder–Davis–Gundy inequality, the Lipschitz continuity of σ and Young's inequality that

$$\mathbb{E}\left[\sup_{s \le t} \left| \int_0^s 1_{\{D_r > 0\}} D_r (\sigma(r, \tilde{X}^1_r) - \sigma(r, \tilde{X}^2_r))\, \mathrm{d}W_r \right| \right] \le C\mathbb{E}\left[\sqrt{\int_0^t 1_{\{D_r > 0\}} D_r^4\, \mathrm{d}r}\right] \le C\mathbb{E}\left[\sqrt{\Gamma_t \int_0^t \Gamma_r\, \mathrm{d}r}\right] \le \frac{1}{2}\mathbb{E}[\Gamma_t] + \frac{1}{2}C^2 \mathbb{E}\left[\int_0^t \Gamma_r\, \mathrm{d}r\right].$$

Putting everything together, we obtain

$$\mathbb{E}[\Gamma_t] \le \Gamma_0 + K \int_0^t \mathbb{E}[\Gamma_r]\, \mathrm{d}r$$

for some constant $K > 0$. Then Gronwall's lemma yields

$$\mathbb{E}\left[\sup_{s \le t} \max(0, \tilde{X}^1_s - \tilde{X}^2_s)^2\right] = \mathbb{E}[\Gamma_t] \le C\Gamma_0 = C \max(0, \xi_1 - \xi_2)^2 \tag{A.3}$$

for some constant $C > 0$. If $\xi_1 \le \xi_2$ this directly yields (iii). For (iv) observe that we have

$$\mathbb{E}\left[\sup_{s \le t} (\tilde{X}^1_s - \tilde{X}^2_s)^2\right] \le \mathbb{E}\left[\sup_{s \le t} \max(0, \tilde{X}^1_s - \tilde{X}^2_s)^2\right] + \mathbb{E}\left[\sup_{s \le t} \max(0, \tilde{X}^2_s - \tilde{X}^1_s)^2\right].$$

Then Inequality (A.3) yields $\mathbb{E}[\sup_{s\le t}(\tilde X_s^1-\tilde X_s^2)^2]\le \tilde C(\xi_1-\xi_2)^2$. The case $p=1$ follows from Jensen's inequality. Claim (v) follows by performing similar arguments with $D=\tilde X^{t,\xi}-X^{t,\xi}$.

To prove Equation (12), we set

$$Y_s=Y_s^{t,\xi}=\int_t^s \mu(u,\tilde X_u^{t,\xi})\,\mathrm{d}u+\int_t^r \sigma(u,\tilde X_{u-}^{t,\xi})\,\mathrm{d}W_u,\qquad \hat l_s=\sup_{t\le r\le s}(\xi+Y_r-b(r))^+ \tag{A.4}$$

and $\hat X=\xi+Y_s-\hat l_s$. Then it is straightforward to show that $(\hat X,\hat l)$ is a solution to the Skorokhod problem associated with Y and barrier b (cf. Slominski and Wojciechowski [43, Definition 2.5]). Since $(\tilde X,l)$ is also a solution, we obtain Equation (12) by uniqueness of solutions to the Skorokhod problem (cf. Slominski and Wojciechowski [43, Proposition 2.4]).

Finally we prove Claim (vi). To this end let $x\le b(t)\wedge b(t-)$ and $t_n\nearrow t$ and $x_n\to x$ as $n\to\infty$. We write $\tilde X^n=\tilde X^{t_n,x_n\wedge b(t_n)}$ and $Y^n=Y^{t_n,x_n\wedge b(t_n)}$ (see Equation (A.4) for the definition of Y). Then we have

$$\begin{aligned}|\tilde X_t^n-x|&=\Big|x_n\wedge b(t_n)-x+Y_t^n-\sup_{t_n\le r\le t}(x_n\wedge b(t_n)+Y_r^n-b(r))^+\Big|\\&\le |x_n\wedge b(t_n)-x|+\sup_{t_n\le r\le t}(x_n\wedge b(t_n)-b(r))^+ +2\sup_{t_n\le r\le t}|Y_r^n|.\end{aligned}$$

Squaring this inequality and taking expectations yields

$$\mathbb{E}[|\tilde X_t^n-x|^2]\le 3|x_n\wedge b(t_n)-x|^2+3\sup_{t_n\le r\le t}((x_n\wedge b(t_n)-b(r))^+)^2+6\mathbb{E}\Big[\sup_{t_n\le r\le t}|Y_r^n|^2\Big].$$

The first two terms converge to 0 for $n\to\infty$ since b is càdlàg and $x\le b(t)\wedge b(t-)$. Regarding the last term, observe that Jensen's and the Burkholder–Davis–Gundy inequality yield

$$\mathbb{E}\Big[\sup_{t_n\le r\le t}|Y_r^n|^2\Big]\le C\int_{t_n}^t \mathbb{E}[\mu(s,\tilde X_s^n)^2+\sigma(s,\tilde X_s^n)^2]\,\mathrm{d}s$$

for some constant $C>0$ (not depending on n). It remains to prove that $\mathbb{E}[\mu(s,\tilde X_s^n)^2+\sigma(s,\tilde X_s^n)^2]$ is a bounded sequence. To this end assume without loss of generality that $\tilde X^0=\tilde X^{0,b(0)}$, then the linear growth of μ and σ, the Markov property of $\tilde X^0$ and Claim (iv) imply

$$\mathbb{E}[\mu(s,\tilde X_s^n)^2+\sigma(s,\tilde X_s^n)^2]\le C_1(1+\mathbb{E}[(\tilde X_s^n-\tilde X_s^0)^2+(\tilde X_s^0)^2])\le C_2(1+\mathbb{E}[(\tilde X_{t_n}^0-x_n\wedge b(t_n))^2+(\tilde X_s^0)^2])$$

for some $C_1,C_2>0$. This is a bounded sequence by Claim (i), which yields $\mathbb{E}[\sup_{t_n\le r\le t}|Y_r^n|^2]\to 0$ as $n\to\infty$. □

Endnotes

[1] See also Proposition 7.

[2] All our results hold analogously for a lower stopping boundary $A_t=(-\infty,b(t)]$ if we impose instead of our single crossing condition that $x\mapsto f(t,x)+(\partial_t+\mathscr{L})g(t,x)$ is nondecreasing.

[3] See Definition 5.

[4] Here and in the sequel we use the notation $\pi(t+)=\lim_{\epsilon\searrow 0}\pi(t+\epsilon)$ and $\pi(t-)=\lim_{\epsilon\searrow 0}\pi(t-\epsilon)$ for the one-sided limits.

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